

RADIAL SOLUTIONS TO THE CAUCHY PROBLEM FOR $\square_{1+3}U = 0$ AS LIMITS OF EXTERIOR SOLUTIONS

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ABSTRACT. We consider the strategy of realizing the solution of a Cauchy problem with radial data as a limit of radial solutions to initial-boundary value problems posed on the exterior of vanishing balls centered at the origin. The goal is to gauge the effectiveness of this approach in a simple, concrete setting: the 3-dimensional, linear wave equation $\square_{1+3}U = 0$ with radial Cauchy data $U(0, x) = \Phi(x) = \varphi(|x|)$, $U_t(0, x) = \Psi(x) = \psi(|x|)$.

We are primarily interested in this as a model situation for other, possibly nonlinear, equations where neither formulae nor abstract existence results are available for the radial symmetric Cauchy problem. In treating the 3-d wave equation we therefore insist on robust arguments based on energy methods and strong convergence. (In particular, this work does not address what can be established via solution formulae.)

Our findings for the 3-d wave equation show that while one can obtain existence of radial Cauchy solutions via exterior solutions, one should not expect such results to be optimal. The standard existence result for the linear wave equation guarantees a unique solution in $C([0, T]; H^s(\mathbb{R}^3))$ whenever $(\Phi, \Psi) \in H^s \times H^{s-1}(\mathbb{R}^3)$. However, within the constrained framework outlined above, we obtain strictly lower regularity for solutions obtained as limits of exterior solutions. We also show that external Neumann solutions yield better regularity than external Dirichlet solutions. Specifically, for Cauchy data in $H^2 \times H^1(\mathbb{R}^3)$ we obtain H^1 -solutions via exterior Neumann solutions, and only L^2 -solutions via exterior Dirichlet solutions.

Key words: Cauchy problem, radial solutions, exterior solutions, Neumann and Dirichlet conditions, Hardy's inequality.

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Notation 1. We use the notations $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}_0^+ = [0, \infty)$. Also, $C_c^\infty(\Omega)$ denotes the set of test functions on an open set Ω , i.e. infinitely smooth functions with compact support contained in Ω . For function of time and spatial position, the time variable t is always listed first, and the spatial variable (x or r) is listed last. Ditto for spaces of such functions. We indicate by subscript “rad” that the functions under consideration are spherically symmetric, e.g. $H_{\text{rad}}^2(\mathbb{R}^3)$ denotes the set of $H^2(\mathbb{R}^3)$ -functions Φ with the property that $\Phi(x) = \varphi(|x|)$ for some function $\varphi : \mathbb{R}_0^+ \rightarrow \mathbb{R}$. We write $H^2 \times H^1(\mathbb{R}^3)$ for $H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$.

Throughout we fix $T > 0$ and $c > 0$ and set

$$\square_{1+1} := \partial_t^2 - c^2 \partial_r^2, \quad \square_{1+3} := \partial_t^2 - c^2 \Delta,$$

where Δ is the 3-d Laplacian. We write ∂_i for ∂_{x_i} . The open ball of radius r about the origin in \mathbb{R}^3 is denoted B_r . We set $\bar{c} = \text{area}(B_1)$ and $\bar{\omega} = \text{volume}(B_1)$. The first standard basis vector in \mathbb{R}^3 is denoted \vec{e}_1 . We write “ $x \lesssim y$ ” to mean that “ $x \leq C \cdot y$ ” for some number C that may depend on fixed parameters (e.g. c and T) and fixed (e.g. cutoff) functions.

1. INTRODUCTION

Establishing global existence of solutions to the Cauchy problem for evolutionary PDEs is a challenging task, especially in several space dimensions and for nonlinear problems. A fairly common situation is that a one-dimensional (1-d) theory is in place, while any extension to several space dimensions raises hard issues. Examples are provided by compressible flow (inviscid or viscous, isentropic or not), general nonlinear hyperbolic conservation laws, and various nonlinear wave equations.

Equations derived as physical models are often rotationally invariant. In such cases it is of interest to consider *radial* (i.e. spherically symmetric \equiv rotationally invariant) solutions that depend on the spatial variable x only through its norm $r = |x|$. Such a reduction yields a 1-d, or more precisely a *quasi*-1-d, problem. As the compressible Euler system makes painfully clear, to establish existence of radial flows it is not sufficient to know how to solve 1-d Cauchy problems. There are two reasons for this: the radial problem is really a mixed initial-boundary value problem (boundary conditions must be prescribed at the origin), and the radial equations will contain geometric source terms that blow up at $r = 0$.

Given the lack of readily available alternatives, it is reasonable to ask if radial problems can be handled via a (truly) 1-d approach where one seeks multi-d, radial Cauchy solutions as limits of approximate, exterior radial solutions. That is, for radial Cauchy data we first solve a corresponding initial-boundary value problem on the exterior of a small ball B_ε centered at the origin. We then want to show that these exterior solutions u_ε converge to a bona fide Cauchy solution as $\varepsilon \downarrow 0$. It is part of the problem to choose boundary conditions for the exterior solutions u_ε along $|x| = \varepsilon$, and to describe how the initial data for u_ε are to be generated from the original Cauchy data.

Our objective in this work is to gauge the effectiveness of this scheme for a case where “everything is known:” the 3-d, linear wave equation with radial data. For a fixed time

$T > 0$, we consider the Cauchy problem:

$$(CP) \quad \begin{cases} \square_{1+3} U = 0 & \text{on } (0, T) \times \mathbb{R}^3 \\ U(0, x) = \Phi(x) & \text{on } \mathbb{R}^3 \\ U_t(0, x) = \Psi(x) & \text{on } \mathbb{R}^3, \end{cases}$$

with radial initial data (Φ, Ψ) .

Of course, in this case one could employ formulae for both the Cauchy problem and the exterior problems, and calculate exactly how exterior solutions converge (or not) to a Cauchy solution. While this is of interest in its own right (we are not aware of a reference) we are here interested in exploring what this model case can tell us about situations where no formulae are available. We shall therefore attempt to “work with one arm tied” and insist on arguments that do not exploit formulae or special properties of the linear 3-d wave equation beyond conservation of energy¹. Also, while weak convergence would suffice to establish existence of a weak solution to the linear wave equation, strong convergence is required for nonlinear problems. We therefore concentrate on strong convergence of exterior solutions.

2. RESULTS AND DISCUSSION

2.1. Main results. We recall the standard existence result for (CP) which guarantees a unique solution $U \in C([0, T]; H^s(\mathbb{R}^3))$ whenever $(\Phi, \Psi) \in H^s \times H^{s-1}(\mathbb{R}^3)$ (any $s \in \mathbb{R}$); see [14]. A natural goal would be a proof of this result (for radial data) via exterior solutions. However, we shall see that the convergence of exterior solutions to the solution of (CP) depends on both

- the regularity of the Cauchy data (Φ, Ψ) , and
- the choice of boundary conditions for the exterior approximations.

For concreteness we consider Cauchy data in $H_{rad}^2 \times H_{rad}^1(\mathbb{R}^3)$ and in $H_{rad}^1 \times L_{rad}^2(\mathbb{R}^3)$. Only in the former case have we been able to establish existence of a solution to (CP) via exterior solutions. Furthermore, exterior Neumann solutions yield only an $H^1(\mathbb{R}^3)$ -solution for (CP), while exterior Dirichlet solutions yield an $L^2(\mathbb{R}^3)$ -solution only (always with the understanding that we avoid solution formulae). Thus, even in the case where we obtain a limiting Cauchy solution, and regardless of the boundary condition we use for the exterior solutions, we are only able to establish strictly less regularity than what is known to hold for the Cauchy solution.

We proceed to give a precise description of our results. We fix radial initial data

$$\Phi(x) = \varphi(|x|) \quad \text{and} \quad \Psi(x) = \psi(|x|),$$

for given functions $\varphi, \psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}$.

Definition 2.1. $U \in C([0, T]; H^1(\mathbb{R}^3))$ is a weak H^1 -solution of (CP) provided

$$\int_0^T \int_{\mathbb{R}^3} U V_{tt} + c^2 \nabla U \cdot \nabla V \, dx dt + \int_{\mathbb{R}^3} \Phi(x) V_t(0, x) - \Psi(x) V(0, x) \, dx = 0 \quad (2.1)$$

whenever $V \in C_c^\infty((-\infty, T) \times \mathbb{R}^3)$.

¹For the convergence of exterior Dirichlet solutions we have found it necessary to exploit the relationship between radial 3-d solutions and 1-d solutions; see discussion in Section 4.3.2 below.

Definition 2.2. $U \in C([0, T]; L^2(\mathbb{R}^3))$ is a weak L^2 -solution of (CP) provided

$$\int_0^T \int_{\mathbb{R}^3} U \square_{1+3} V \, dx dt + \int_{\mathbb{R}^3} \Phi(x) V_t(0, x) - \Psi(x) V(0, x) \, dx = 0 \quad (2.2)$$

whenever $V \in C_c^\infty((-\infty, T) \times \mathbb{R}^3)$.

Of course, a weak H^1 -solution is automatically a weak L^2 -solution.

We next describe how the initial data and the solutions for the exterior problems are generated. For any given sequence of vanishing radii $\varepsilon_n \downarrow 0$ we construct smooth, radial initial data (Φ_n, Ψ_n) for the exterior problems by suitably cutting off and mollifying the original Cauchy data (Φ, Ψ) . The existence and regularity of the corresponding exterior solutions $U_n(t, x)$ on $|x| > \varepsilon_n$ is a genuine 1-d issue and is taken for granted. The U_n will be smooth and satisfy the boundary conditions in a classical sense. At each time t we extend $U_n(t, \cdot)$ in a continuous manner to a function $\tilde{U}_n(t, \cdot)$ defined on all of \mathbb{R}^3 : for Neumann solutions we take $\tilde{U}_n(t, \cdot)$ to be constant equal to $U_n(t, \varepsilon_n \vec{e}_1)$ on B_{ε_n} , while for Dirichlet solutions $\tilde{U}_n(t, \cdot)$ is defined to vanish identically on B_{ε_n} .

We then want to argue that the extensions $\tilde{U}_n(t, x)$ converge to a function $U(t, x)$, and that this U is a bona fide weak solution of the original Cauchy problem (CP) according to one of the definitions above. Our “positive” findings are summarized in the following theorem.

Theorem 2.3. *Consider the Cauchy problem (CP) for the linear wave equation in three space dimensions. Let $\varepsilon_n \downarrow 0$ be any sequence of vanishing radii and consider the sequences U_n^N and U_n^D of exterior solutions on $\mathbb{R}^3 \setminus B_{\varepsilon_n}$ satisfying vanishing Neumann and vanishing Dirichlet conditions, respectively, on $|x| = \varepsilon_n$. The functions obtained by extending these continuously at each time as constants on B_{ε_n} are denoted \tilde{U}_n^N and \tilde{U}_n^D , respectively. Then, with initial data for (CP) belonging to $H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, we have that*

- (i) *the sequence $(\tilde{U}_n^N)_n$ converges in $C([0, T]; H^1(\mathbb{R}^3))$ to a weak H^1 -solution of (CP) according to Definition 2.1.*
- (ii) *a subsequence of $(\tilde{U}_n^D)_n$ converges in $C([0, T]; L^2(\mathbb{R}^3))$ to a weak L^2 -solution of (CP) according to Definition 2.2.*

The details of the arguments for part (i) and part (ii) of Theorem 2.3 are given in Section 3 and Section 4, respectively.

Before carrying out the details of the proof we make some remarks. First, while Theorem 2.3 does provide an existence result for (CP), the more important aspect in our view concerns what it does *not* provide. Specifically:

- (1) we are not able to reproduce the standard existence result for (CP), according to which the solution with $H^2 \times H^1(\mathbb{R}^3)$ -data belongs to $C([0, T]; H^2(\mathbb{R}^3))$; and
- (2) our analysis requires at least $H^2 \times H^1(\mathbb{R}^3)$ -data for (CP): we are not able to carry out a similar analysis for $H^1 \times L^2(\mathbb{R}^3)$ -data. See Remarks 3.3 and 3.4.

These issues are directly related to our insistence that the proof should be based on energy methods and strong convergence, and thus in principle be applicable to other, possibly non-linear, situations. As noted earlier, (1) and (2) highlight the shortcomings of the approach of realizing solutions of initial value problems as limits of exterior initial-boundary value problems.

Another point is that we obtain a less regular limit from exterior Dirichlet solutions than from exterior Neumann solutions. To see that this is reasonable, recall that solutions

of (CP) may contain large amplitudes and gradients near the origin due to focussing of waves. Now, the value at $r = \varepsilon_n$ of an exterior Neumann solution on $\mathbb{R}_t \times (\mathbb{R}^3 \setminus B_{\varepsilon_n})$ is “free to move.” Thus, Neumann solutions can incorporate large amplitudes near the origin and accurately mimic the behavior of the solution of the Cauchy problem. It is therefore reasonable to expect that exterior Neumann solutions approximate solutions of (CP) accurately, and indeed converge to such as $\varepsilon_n \downarrow 0$.

On the other hand, for an exterior Dirichlet solution the value at $r = \varepsilon_n$ is “pinned down” to vanish. This introduces *additional*, large gradients in the approximate solutions near the origin - a situation clearly less favorable for convergence. This difference between Neumann and Dirichlet conditions will be evident from the analysis of the exterior data generated from the initial data for (CP). The technical reason for the difference between the two cases is that, while the set of $C_c^\infty(\mathbb{R}^3)$ -functions that are *constant* on some ball about the origin is dense in $H^2(\mathbb{R}^3)$, the set of $C_c^\infty(\mathbb{R}^3)$ -functions that *vanish* on some ball about the origin is not. To show these facts we make use of Hardy’s inequality in \mathbb{R}^3 . For completeness we include the relevant statements in Section 2.3 below.

As remarked above we focus on arguments that provide strong convergence. In the case of the 3-d wave equation this can be accomplished in different ways. For Neumann exterior solutions we shall argue via completeness in $C([0, T]; H^1(\mathbb{R}^3))$. Alternatively we could have argued by strong compactness in the same space. On the other hand, for the case of exterior Dirichlet solutions, we have been able to establish convergence to a weak L^2 -solution only via strong compactness in $C([0, T]; L^2(\mathbb{R}^3))$. Furthermore, for the latter case, while avoiding explicit solution formulae we have found it necessary to exploit the fact that radial solutions U of the 3-d wave equation correspond to solutions $u = rU$ of the 1-d wave equation (see Section 4.3.2).

We observe that $H^1(\mathbb{R}^3)$ contains unbounded functions (e.g., $|x|^{\delta-\frac{1}{2}}$ for $0 < \delta < \frac{1}{2}$), while $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$. Thus the result above covers cases with unbounded initial data for $\partial_t U$. Finally, we note that as far as existence of a solution to (CP) is concerned, it would suffice to establish convergence of exterior solutions for a single sequence of vanishing radii ε_n . However, we can treat arbitrary sequences of vanishing radii without much extra effort; see Remark 3.2.

2.2. Related works. The scheme of generating radial solutions to Cauchy problems as limits of exterior solutions has been applied to various models for fluid flow. For the compressible isentropic Navier-Stokes system see Hoff [5]; see also [6, 10].

For compressible Euler flows already the exterior problem is highly challenging. The exterior problem for radial, isothermal Euler solutions was analyzed by Makino, Mizohata, and Ukai in [11, 12]. Their work is formulated in a BV setting and exploits the fact that a particular feature of 1-d isothermal gas-dynamics (translation invariance of wave curves) makes it possible to treat large data (this was first observed by Nishida [13]). However, to the best of our knowledge their results have not been extended to the radial Cauchy problem via a limiting procedure as studied in the present paper. Recently, Chen and Perepelitsa have studied this problem via compensated compactness and a combination of vanishing viscosity and exterior solutions; see [3] and references therein for further details.

For incompressible flow there is a considerable literature on vanishing obstacle problems, and more precise information is available. There are cases of 2-d incompressible Euler flow where the limit of exterior solutions corresponding to a sequence of vanishing obstacles does not solve the original system. Instead it satisfies an equation with an additional

forcing term parametrized by the vorticity of the initial data; see [8] for details. The corresponding analysis for 2-d viscous, incompressible flow was treated in [9] and showed that the only lasting effect of the obstacle on the limit solution is to add a δ -function to the initial vorticity. On the other hand, for purely radial flows the limit will in any case satisfy the original, unperturbed Cauchy problem. For recent results on incompressible flow and vanishing obstacles, see [7, 16].

Finally, in a somewhat different setting, Rauch and Taylor [15] considered (among several other issues) the wave equation on sequences of domains Ω_n converging to a given domain Ω . Under the condition that the initial data for the unperturbed problem on Ω belong to $C_c^\infty(\Omega_n)$ for all n , they established convergence in energy norm of Dirichlet solutions on Ω_n to the Dirichlet solution on Ω . The condition on the initial data seems to prevent a straightforward adaption of their techniques to the problem of obtaining solutions to Cauchy problems as limits of exterior solutions. We note that [15] exploits solution formulae. As mentioned above, we are deliberately avoiding this in the present work since the wave equation here only serves as a “probe” for other situations where no formulae are available. On the other hand, if we do focus on the wave equation, we expect that much stronger results (possibly optimal regularity for Cauchy solutions) can be obtained via solution formulae. This issue will be pursued in future work.

2.3. Two versions of Hardy’s inequality. For later reference we include the following estimates (taken from Theorem 7 in Section 5.8.4 and Exercise 5.11.16 in [4], respectively). There exists a constant C such that

$$\left\| \frac{u}{|x|} \right\|_{L^2(\mathbb{R}^3)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^3)} \quad \text{whenever } u \in H^1(\mathbb{R}^3), \quad (2.3)$$

and

$$\left\| \frac{u}{|x|} \right\|_{L^2(B_1)} \leq C \|u\|_{H^1(B_1)} \quad \text{whenever } u \in H^1(B_1), \quad (2.4)$$

where B_1 is the unit ball in \mathbb{R}^3 .

3. CAUCHY SOLUTION AS LIMIT OF EXTERIOR NEUMANN SOLUTIONS: WEAK H^1 -SOLUTION FOR $H^2 \times H^1$ -DATA VIA COMPLETENESS

3.1. Exterior Neumann data and solutions. We employ the following scheme for exterior Neumann solutions approximating the solution of the Cauchy problem (CP) with initial data $(\Phi, \Psi) \in H_{rad}^2 \times H_{rad}^1(\mathbb{R}^3)$.

- (1) For each $n \geq 1$ we fix $\bar{\Phi}_n, \bar{\Psi}_n$ in $C_{c,rad}^\infty(\mathbb{R}^3)$ such that

$$\bar{\Phi}_n \rightarrow \Phi \quad \text{in } H^2(\mathbb{R}^3) \quad \text{and} \quad \bar{\Psi}_n \rightarrow \Psi \quad \text{in } H^1(\mathbb{R}^3) \quad \text{as } n \rightarrow \infty. \quad (3.1)$$

For concreteness we do this as follows: let (r_n) and (δ_n) be positive sequences increasing to ∞ and decreasing to 0, respectively, with $r_n > 1 > \delta_n$, and set

$$\begin{aligned} \bar{\Phi}_n &:= (\Phi \cdot \chi_{|x| < r_n}) * \eta_{\delta_n} =: \hat{\Phi}_n * \eta_{\delta_n}, \\ \bar{\Psi}_n &:= (\Psi \cdot \chi_{|x| < r_n}) * \eta_{\delta_n} =: \hat{\Psi}_n * \eta_{\delta_n}, \end{aligned}$$

where η is a standard mollifier and $\eta_{\delta_n}(x) := \frac{1}{\delta_n^3} \eta(\frac{x}{\delta_n})$. These choices guarantee that (3.1) hold. Note that the use of a standard (in particular, radial) mollifier implies that $\bar{\Phi}_n, \bar{\Psi}_n$ are radial. For later reference we record that

$$\|\bar{\Phi}_n\|_{H^2(|x| > s)} \leq \|\Phi\|_{H^2(|x| > s-1)}, \quad \|\bar{\Psi}_n\|_{H^1(|x| > s)} \leq \|\Psi\|_{H^1(|x| > s-1)} \quad (3.2)$$

whenever $s > 1$.

Remark 3.1. *The convergence in (3.1) will hold in any H^k -space that Φ and Ψ belong to. However, we shall next approximate $\bar{\Phi}_n$ and $\bar{\Psi}_n$ with smooth functions Φ_n and Ψ_n satisfying Neumann conditions on the surface of small balls about the origin. This introduces possibly large gradients and Φ_n and Ψ_n will be close to Φ and Ψ only in certain Sobolev spaces.*

- (2) We now fix any sequence (ε_n) with $\varepsilon_n \downarrow 0$. These are the radii of the vanishing balls whose exterior Neumann solutions we want to show converge to the solution of the original Cauchy problem (CP). To smoothly approximate the Cauchy data (Φ, Ψ) for (CP) with exterior Neumann data (Φ_n, Ψ_n) we fix a C^2 -smooth, nondecreasing function $\beta : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ with

$$\beta \equiv 1 \quad \text{on } [0, 1], \quad \beta(s) = s \quad \text{for } s \geq 2. \quad (3.3)$$

For convenience we further require that β satisfies

$$\beta(s) < s, \quad \beta'(s) > 0, \quad |\beta''(s)| \leq C\beta'(s), \quad \text{for all } s \in (1, 2), \quad (3.4)$$

for some positive constant C . (A direct calculation shows that the function

$$\beta(t) := \begin{cases} 1 & 0 \leq t \leq 1 \\ 1 + 6(t-1)^3 - 8(t-1)^4 + 3(t-1)^5 & 1 < t < 2 \\ t & t \geq 2, \end{cases}$$

meets all the requirements.) Then, with

$$\bar{\varphi}_n(|x|) := \bar{\Phi}_n(x) \quad \text{and} \quad \bar{\psi}_n(|x|) := \bar{\Psi}_n(x) \quad x \in \mathbb{R}^3.$$

we define

$$\Phi_n(x) := \bar{\varphi}_n(\varepsilon_n \beta(\frac{|x|}{\varepsilon_n})) \quad \text{and} \quad \Psi_n(x) := \bar{\psi}_n(\varepsilon_n \beta(\frac{|x|}{\varepsilon_n})) \quad x \in \mathbb{R}^3. \quad (3.5)$$

Note that the restrictions of both Φ_n and Ψ_n to the exterior domain

$$\Omega_n := \{x \in \mathbb{R}^3 : |x| \geq \varepsilon_n\}$$

satisfy homogeneous Neumann conditions at $|x| = \varepsilon_n$.

We refer to (Φ_n, Ψ_n) as the *Neumann data* corresponding to the original Cauchy data (Φ, Ψ) for (CP). Note that the Neumann data are defined on all of \mathbb{R}^3 . We analyze their convergence to (Φ, Ψ) in Section 3.2 below.

- (3) It is standard that the exterior Neumann problem on Ω_n with the smooth initial data $(\Phi_n|_{\Omega_n}, \Psi_n|_{\Omega_n})$ has a unique, smooth, and global-in-time solution which we denote by $U_n(t, x)$. This may be established via solution formulae based on d'Alembert's formula for the 1-d linear wave equation, [1, 14]. To compare these we extend each $U_n(t, x)$ continuously as a constant to all of \mathbb{R}^3 at each time:

$$\tilde{U}_n(t, x) := \begin{cases} U_n(t, \varepsilon_n \vec{e}_1) & \text{for } |x| \leq \varepsilon_n \\ U_n(t, x) & \text{for } |x| \geq \varepsilon_n, \end{cases} \quad (3.6)$$

Note that \tilde{U}_n is C^1 -smooth across $|x| = \varepsilon_n$ at each time.

3.2. Neumann data vs. Cauchy data. We next investigate how the Neumann data (Φ_n, Ψ_n) , which are defined on all of \mathbb{R}^3 , approximate the original Cauchy data (Φ, Ψ) , when the latter belongs to $H^2 \times H^1(\mathbb{R}^3)$. This information will be used below to estimate the distance between exterior solutions at later times $t \in [0, T]$ in terms of initial distances. Specifically, we want to estimate

$$\|\Phi_n - \Phi\|_{H^2(\mathbb{R}^3)} \quad \text{and} \quad \|\Psi_n - \Psi\|_{H^1(\mathbb{R}^3)}.$$

Thanks to (3.1) it suffices to estimate

$$\|\Phi_n - \bar{\Phi}_n\|_{H^2(\mathbb{R}^3)} \quad \text{and} \quad \|\Psi_n - \bar{\Psi}_n\|_{H^1(\mathbb{R}^3)}.$$

To lighten the notation we consider, for now, a fixed function $\bar{F} \in C_{c,rad}^\infty(\mathbb{R}^3)$, say

$$\bar{F}(x) = \bar{f}(|x|),$$

and set

$$F_\varepsilon(x) = f_\varepsilon(|x|) := \bar{f}\left(\varepsilon\beta\left(\frac{|x|}{\varepsilon}\right)\right), \quad \varepsilon > 0.$$

Here, ε corresponds to ε_n , \bar{F} corresponds to $\bar{\Phi}_n$ or $\bar{\Psi}_n$, and F_ε corresponds to Φ_n or Ψ_n , respectively. In Sections 3.2.1-3.2.3 below we provide the details for showing that F_ε converges to \bar{F} in $H^2(\mathbb{R}^3)$ as $\varepsilon \downarrow 0$. Before starting we note that

$$\sum_{1 \leq i \leq 3} |\partial_i \bar{F}(x)|^2 = |\bar{f}'(|x|)|^2 \tag{3.7}$$

and

$$\sum_{1 \leq i,j \leq 3} |\partial_{ij} \bar{F}(x)|^2 = |\bar{f}''(|x|)|^2 + \frac{2}{|x|^2} |\bar{f}'(|x|)|^2. \tag{3.8}$$

Remark 3.2. *Consider estimating*

$$\|F_\varepsilon - \bar{F}\|_{L^2(\mathbb{R}^3)}^2 \equiv \int_{|x| < 2\varepsilon} \left| \bar{f}\left(\varepsilon\beta\left(\frac{|x|}{\varepsilon}\right)\right) - \bar{f}(|x|) \right|^2 dx.$$

A straightforward bound would be that

$$\|F_\varepsilon - \bar{F}\|_{L^2(\mathbb{R}^3)}^2 \lesssim \left(\sup_{0 < r < 2\varepsilon} |\bar{f}(r)|^2 \right) \varepsilon^3,$$

with similar estimates holding for $\|\partial_i F_\varepsilon - \partial_i \bar{F}\|_{L^2(\mathbb{R}^3)}^2$ and $\|\partial_{ij} F_\varepsilon - \partial_{ij} \bar{F}\|_{L^2(\mathbb{R}^3)}^2$. However, the coefficients on the right hand sides of these bounds depend on the sup-norms of \bar{F} and its derivatives, i.e. on $\bar{\Phi}_n$ or $\bar{\Psi}_n$ and their derivatives, in our application of these estimates. In order to show (via energy arguments) that the L^2 -distances in question vanish as n increases, we would need to carefully choose the radii ε_n . The final result would yield H^1 -convergence of the corresponding solutions for suitably chosen sequences of radii (ε_n) . We shall see that a slightly more detailed argument based on Sobolev norms will apply to any sequence (ε_n) .

3.2.1. $\|F_\varepsilon - \bar{F}\|_{L^2(\mathbb{R}^3)}$. Splitting the calculation over the two subregions $\{0 < |x| < \varepsilon\}$ and $\{\varepsilon < |x| < 2\varepsilon\}$, reducing to 1-d integrals, employing the Fundamental Theorem of Calculus and the Cauchy-Schwarz inequality, give:

$$\begin{aligned}
\|F_\varepsilon - \bar{F}\|_{L^2(\mathbb{R}^3)}^2 &\equiv \int_{|x| < 2\varepsilon} |f_\varepsilon(|x|) - \bar{f}(|x|)|^2 dx \\
&= \int_{|x| < \varepsilon} |\bar{f}(\varepsilon) - \bar{f}(|x|)|^2 dx + \int_{\varepsilon < |x| < 2\varepsilon} |\bar{f}(|x|) - \bar{f}(\varepsilon\beta(\frac{|x|}{\varepsilon}))|^2 dx \\
&= \bar{c} \int_0^\varepsilon |\bar{f}(\varepsilon) - \bar{f}(r)|^2 r^2 dr + \bar{c} \int_\varepsilon^{2\varepsilon} |\bar{f}(r) - \bar{f}(\varepsilon\beta(\frac{r}{\varepsilon}))|^2 r^2 dr \\
&= \bar{c} \int_0^\varepsilon \left| \int_r^\varepsilon \bar{f}'(\xi) d\xi \right|^2 r^2 dr + \bar{c} \int_\varepsilon^{2\varepsilon} \left| \int_{\varepsilon\beta(\frac{r}{\varepsilon})}^r \bar{f}'(\xi) d\xi \right|^2 r^2 dr \\
&\lesssim \int_0^\varepsilon (\varepsilon - r) \left[\int_r^\varepsilon |\bar{f}'(\xi)|^2 d\xi \right] r^2 dr + \int_\varepsilon^{2\varepsilon} (r - \varepsilon\beta(\frac{r}{\varepsilon})) \left[\int_{\varepsilon\beta(\frac{r}{\varepsilon})}^r |\bar{f}'(\xi)|^2 d\xi \right] r^2 dr \\
&\lesssim \int_0^\varepsilon (\varepsilon - r) \left[\int_r^\varepsilon |\bar{f}'(\xi)|^2 \xi^2 d\xi \right] dr + \int_\varepsilon^{2\varepsilon} \frac{r^3}{(\varepsilon\beta(\frac{r}{\varepsilon}))^2} \left[\int_{\varepsilon\beta(\frac{r}{\varepsilon})}^r |\bar{f}'(\xi)|^2 \xi^2 d\xi \right] dr \\
&\lesssim \varepsilon^2 \|\nabla \bar{F}\|_{L^2(B_\varepsilon)}^2 + \varepsilon^2 \|\nabla \bar{F}\|_{L^2(B_{2\varepsilon} \setminus B_\varepsilon)}^2 \lesssim \varepsilon^2 \|\nabla \bar{F}\|_{L^2(B_{2\varepsilon})}^2. \tag{3.9}
\end{aligned}$$

3.2.2. $\|\nabla F_\varepsilon - \nabla \bar{F}\|_{L^2(\mathbb{R}^3)}$. We apply (3.7) and make the same split as above. However, we now treat $\bar{f}'(|x|)$ and $\bar{f}'(\varepsilon\beta(\frac{|x|}{\varepsilon}))$ separately and use that $\beta'(s)$ vanishes for $0 < s < 1$, to get:

$$\begin{aligned}
\sum_i \|\partial_i F_\varepsilon - \partial_i \bar{F}\|_{L^2(\mathbb{R}^3)}^2 &= \int_{|x| < 2\varepsilon} \left| \bar{f}'(\varepsilon\beta(\frac{|x|}{\varepsilon}))\beta'(\frac{|x|}{\varepsilon}) - \bar{f}'(|x|) \right|^2 dx \\
&\lesssim \int_{\varepsilon < |x| < 2\varepsilon} |\bar{f}'(\varepsilon\beta(\frac{|x|}{\varepsilon}))\beta'(\frac{|x|}{\varepsilon})|^2 dx + \int_{|x| < 2\varepsilon} |\bar{f}'(|x|)|^2 dx \\
&\lesssim \int_{\varepsilon < |x| < 2\varepsilon} |\bar{f}'(\varepsilon\beta(\frac{|x|}{\varepsilon}))|^2 \beta'(\frac{|x|}{\varepsilon}) dx + \|\nabla \bar{F}\|_{L^2(B_{2\varepsilon})}^2. \tag{3.10}
\end{aligned}$$

For the last integral we reduce to 1-d and use the change of variable $\xi = \varepsilon\beta(\frac{r}{\varepsilon})$:

$$\begin{aligned}
\int_{\varepsilon < |x| < 2\varepsilon} |\bar{f}'(\varepsilon\beta(\frac{|x|}{\varepsilon}))|^2 \beta'(\frac{|x|}{\varepsilon}) dx &\lesssim \int_\varepsilon^{2\varepsilon} |\bar{f}'(\varepsilon\beta(\frac{r}{\varepsilon}))|^2 \beta'(\frac{r}{\varepsilon}) r^2 dr \\
&\lesssim \varepsilon^2 \int_\varepsilon^{2\varepsilon} |\bar{f}'(\varepsilon\beta(\frac{r}{\varepsilon}))|^2 \beta'(\frac{r}{\varepsilon}) dr = \varepsilon^2 \int_\varepsilon^{2\varepsilon} |\bar{f}'(\xi)|^2 d\xi \\
&\lesssim \int_\varepsilon^{2\varepsilon} |\bar{f}'(\xi)|^2 \xi^2 d\xi \lesssim \int_{\varepsilon < |x| < 2\varepsilon} |\bar{f}'(|x|)|^2 dx \\
&\leq \|\nabla \bar{F}\|_{L^2(B_{2\varepsilon})}^2.
\end{aligned}$$

Using this in (3.10) gives

$$\sum_i \|\partial_i F_\varepsilon - \partial_i \bar{F}\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla \bar{F}\|_{L^2(B_{2\varepsilon})}. \tag{3.11}$$

3.2.3. $\|D^2 F_\varepsilon - D^2 \bar{F}\|_{L^2(\mathbb{R}^3)}$. For $\bar{F} \in H^2(\mathbb{R}^3)$ we proceed similarly. Applying (3.8), (3.4)₃, that $\beta'(s)$ and $\beta''(s)$ both vanish for $s < 1$, reducing to 1-d, and employing the same change of variables as above, yield

$$\begin{aligned}
& \sum_{i,j} \|\partial_{ij} F_\varepsilon - \partial_{ij} \bar{F}\|_{L^2(\mathbb{R}^3)}^2 \\
& \lesssim \int_{\varepsilon < |x| < 2\varepsilon} |\bar{f}''(\varepsilon\beta(\frac{|x|}{\varepsilon}))|^2 |\beta'(\frac{|x|}{\varepsilon})|^4 + |\bar{f}'(\varepsilon\beta(\frac{|x|}{\varepsilon}))|^2 |\beta'(\frac{|x|}{\varepsilon})|^2 \frac{1}{|x|^2} \\
& \quad + \frac{1}{\varepsilon^2} |\bar{f}'(\varepsilon\beta(\frac{|x|}{\varepsilon}))|^2 |\beta''(\frac{|x|}{\varepsilon})|^2 dx + \sum_{i,j} \int_{B_{2\varepsilon}} |\partial_{ij} \bar{F}(x)|^2 dx \\
& \lesssim \int_{\varepsilon < |x| < 2\varepsilon} \left(|\bar{f}''(\varepsilon\beta(\frac{|x|}{\varepsilon}))|^2 + \frac{1}{\varepsilon^2} |\bar{f}'(\varepsilon\beta(\frac{|x|}{\varepsilon}))|^2 \right) \beta'(\frac{|x|}{\varepsilon}) dx + \|\bar{F}\|_{H^2(B_{2\varepsilon})}^2 \\
& \lesssim \varepsilon^2 \int_{\varepsilon}^{2\varepsilon} \left(|\bar{f}''(\varepsilon\beta(\frac{r}{\varepsilon}))|^2 + \frac{1}{\varepsilon^2} |\bar{f}'(\varepsilon\beta(\frac{r}{\varepsilon}))|^2 \right) \beta'(\frac{r}{\varepsilon}) dr + \|\bar{F}\|_{H^2(B_{2\varepsilon})}^2 \\
& \lesssim \int_{\varepsilon}^{2\varepsilon} \left(|\bar{f}''(\xi)|^2 + \frac{2}{\xi^2} |\bar{f}'(\xi)|^2 \right) \xi^2 d\xi + \|\bar{F}\|_{H^2(B_{2\varepsilon})}^2 \\
& \lesssim \int_{\varepsilon < |x| < 2\varepsilon} \left(|\bar{f}''(|x|)|^2 + \frac{2}{|x|^2} |\bar{f}'(|x|)|^2 \right) dx + \|\bar{F}\|_{H^2(B_{2\varepsilon})}^2 \lesssim \|\bar{F}\|_{H^2(B_{2\varepsilon})}^2.
\end{aligned}$$

As $\bar{F} \in H^2(\mathbb{R}^3)$, Hardy's inequality (2.3) applies and we have

$$\frac{1}{\varepsilon} \|\nabla \bar{F}\|_{L^2(B_{2\varepsilon})} \lesssim \left\| \frac{\nabla \bar{F}}{|x|} \right\|_{L^2(B_{2\varepsilon})} \leq \left\| \frac{\nabla \bar{F}}{|x|} \right\|_{L^2(\mathbb{R}^3)} \lesssim \|\bar{F}\|_{H^2(\mathbb{R}^3)}.$$

Using this in (3.9) and (3.11), we get that

$$\|F_\varepsilon - \bar{F}\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^2 \|\bar{F}\|_{H^2(\mathbb{R}^3)}, \quad (3.12)$$

$$\|F_\varepsilon - \bar{F}\|_{H^1(\mathbb{R}^3)} \lesssim \varepsilon \|\bar{F}\|_{H^2(\mathbb{R}^3)}, \quad (3.13)$$

while

$$\|F_\varepsilon - \bar{F}\|_{H^2(\mathbb{R}^3)} \lesssim \|\bar{F}\|_{H^2(B_{2\varepsilon})}, \quad (3.14)$$

We finally apply these estimates to the cases $(\bar{F}, F_\varepsilon) = (\bar{\Phi}_n, \Phi_n)$ and $(\bar{F}, F_\varepsilon) = (\bar{\Psi}_n, \Psi_n)$. Recalling (3.1) we conclude that the Neumann data (Φ_n, Ψ_n) corresponding to the original Cauchy data $(\Phi, \Psi) \in H^2 \times H^1(\mathbb{R}^3)$ for (CP), satisfy

$$\Phi_n \rightarrow \Phi \quad \text{in } H^2(\mathbb{R}^3) \quad (3.15)$$

and

$$\Psi_n \rightarrow \Psi \quad \text{in } H^1(\mathbb{R}^3). \quad (3.16)$$

In particular, as noted in Section 2, this establishes that the set

$$\{u \in C_{c,rad}^\infty(\mathbb{R}^3) \mid u \text{ is constant on } B_r \text{ for some } r > 0\}$$

is dense in $H_{rad}^2(\mathbb{R}^3)$. As we shall see in Section 4.2, the situation is less favorable when (Φ, Ψ) is approximated by data satisfying Dirichlet boundary conditions.

3.3. Convergence of exterior Neumann solutions via completeness. We proceed to analyze the convergence of the exterior Neumann solution $U_n(t)$ (really, their extensions $\tilde{U}_n(t)$) to a solution of the original Cauchy problem (CP).

Notwithstanding the convergence of the data recorded in (3.15)-(3.16), we are able to establish only H^1 -convergence of the corresponding solutions. In particular, by restricting ourselves to energy estimates for exterior solutions, we are not able to reproduce the optimal H^2 -regularity for the Cauchy solution (CP) with (radial) data in $H^2 \times H^1(\mathbb{R}^3)$.

In the remainder of this section we give the details of a convergence argument based on completeness in $C([0, T]; H^1(\mathbb{R}^3))$. The proof that the limit is indeed a weak H^1 -solution is given in Section 3.3.4. A similar, and somewhat simpler, approach employing strong compactness in H^1 would give convergence along a subsequence. We consider this latter type of argument below for exterior Dirichlet solutions; see Section 4.3.1.

3.3.1. 1st and 2nd order energies. We shall consider how the energy of the difference between $\tilde{U}_n(t)$ and $\tilde{U}_m(t)$ changes in time. Fix $n > m$ such that $\varepsilon_n < \varepsilon_m$, and set

$$Z_{m,n} := \tilde{U}_n - \tilde{U}_m.$$

We then define the first order energy

$$\mathcal{E}_{m,n}(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t Z_{m,n}(t, x)|^2 + c^2 |\nabla Z_{m,n}(t, x)|^2 dx, \quad (3.17)$$

and the second order energy

$$m_{,n}(t) := \sum_{i=1}^3 \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t \partial_i Z_{m,n}(t, x)|^2 + c^2 |\nabla \partial_i Z_{m,n}(t, x)|^2 dx. \quad (3.18)$$

Note that the $m_{,n}$ are well-defined since the U_n are C^2 -smooth functions satisfying the Neumann condition at $|x| = \varepsilon_n$. Next, for any function $W(t, x)$ defined and weakly differentiable on $\mathbb{R} \times \{|x| > \varepsilon\}$ for some $\varepsilon > 0$, we define

$$\mathcal{E}_W(t) := \frac{1}{2} \int_{|x| > \varepsilon} |\partial_t W(t, x)|^2 + c^2 |\nabla W(t, x)|^2 dx,$$

and

$$w(t) := \sum_{i=1}^3 \mathcal{E}_{\partial_i W}(t) = \sum_{i=1}^3 \frac{1}{2} \int_{|x| > \varepsilon} |\partial_t \partial_i W(t, x)|^2 + c^2 |\nabla \partial_i W(t, x)|^2 dx.$$

We observe that $\mathcal{E}_{m,n}(t)$ is calculated over all of \mathbb{R}^3 while $\mathcal{E}_W(t)$ is calculated over the exterior of a ball; similarly for $m_{,n}(t)$ and $w(t)$.

Since U_n solves the wave equation with a vanishing Neumann boundary condition, $\mathcal{E}_{U_n}(t)$ is constant in time. The second order energy $m_{,n}(t)$ is the sum of the 1st order energies associated with the partial derivatives $\partial_i U_n$. These are again smooth solutions of the wave equation, and a direct calculation shows that also $U_n(t)$ is conserved for *radial* exterior Neumann solutions. (This is not necessarily the case for non-radial solutions.)

3.3.2. Cauchy property in H^1 at fixed times. In this section we show that the $\tilde{U}_n(t)$ form a Cauchy sequence in $H^1(\mathbb{R}^3)$ at each fixed time. We proceed to estimate how the energy difference $\mathcal{E}_{m,n}(t)$ in (3.17) evolves in time. The result will then be used to estimate the L^2 -distance between \tilde{U}_n and \tilde{U}_m ; see Step 4 below. We first rewrite $\mathcal{E}_{m,n}(t)$ by considering

the contributions from $\{|x| < \varepsilon_n\}$, $\{\varepsilon_n < |x| < \varepsilon_m\}$, and $\{|x| > \varepsilon_m\}$. To lighten the notation we set

$$u_n(t, r) := U_n(t, r\vec{e}_1).$$

We have:

$$\begin{aligned} \mathcal{E}_{m,n}(t) &= \frac{1}{2} \int_{|x| < \varepsilon_n} |\partial_t U_n(t, \varepsilon_n \vec{e}_1) - \partial_t U_m(t, \varepsilon_m \vec{e}_1)|^2 dx \\ &+ \frac{1}{2} \int_{\varepsilon_n < |x| < \varepsilon_m} |\partial_t U_n(t, x) - \partial_t U_m(t, \varepsilon_m \vec{e}_1)|^2 + c^2 |\nabla U_n(t, x)|^2 dx \\ &+ \frac{1}{2} \int_{|x| > \varepsilon_m} |\partial_t [U_n(t, x) - U_m(t, x)]|^2 + c^2 |\nabla [U_n(t, x) - U_m(t, x)]|^2 dx \\ &= \left\{ \frac{1}{2} \bar{\omega} \varepsilon_n^3 |\partial_t u_n(t, \varepsilon_n) - \partial_t u_m(t, \varepsilon_m)|^2 + \frac{1}{2} \bar{\omega} (\varepsilon_m^3 - \varepsilon_n^3) |\partial_t u_m(t, \varepsilon_m)|^2 \right\} \\ &\quad - \partial_t u_m(t, \varepsilon_m) \cdot \int_{\varepsilon_n < |x| < \varepsilon_m} \partial_t U_n(t, x) dx + \frac{1}{2} \int_{\varepsilon_n < |x| < \varepsilon_m} |\partial_t U_n(t, x)|^2 + c^2 |\nabla U_n(t, x)|^2 dx \\ &\quad + \frac{1}{2} \int_{|x| > \varepsilon_m} |\partial_t [U_n(t, x) - U_m(t, x)]|^2 + c^2 |\nabla [U_n(t, x) - U_m(t, x)]|^2 dx. \quad (3.19) \end{aligned}$$

Differentiating in time yields

$$\begin{aligned} \dot{\mathcal{E}}_{m,n}(t) &= \frac{d}{dt} \left\{ \frac{1}{2} \bar{\omega} \varepsilon_n^3 |\partial_t u_n(t, \varepsilon_n) - \partial_t u_m(t, \varepsilon_m)|^2 + \frac{1}{2} \bar{\omega} (\varepsilon_m^3 - \varepsilon_n^3) |\partial_t u_m(t, \varepsilon_m)|^2 \right\} \\ &\quad - \partial_{tt} u_m(t, \varepsilon_m) \cdot \int_{\varepsilon_n < |x| < \varepsilon_m} \partial_t U_n(t, x) dx - \partial_t u_m(t, \varepsilon_m) \cdot \int_{\varepsilon_n < |x| < \varepsilon_m} \partial_{tt} U_n(t, x) dx \\ &\quad + \int_{\varepsilon_n < |x| < \varepsilon_m} (\partial_t U_n) (\partial_{tt} U_n) + c^2 \nabla U_n \cdot \nabla (\partial_t U_n) dx \\ &\quad + \int_{|x| > \varepsilon_m} (\partial_t [U_n - U_m]) (\partial_{tt} [U_n - U_m]) + c^2 \nabla [U_n - U_m] \cdot \nabla [\partial_t U_n - \partial_t U_m] dx. \end{aligned}$$

As U_n and U_m solve the wave equation on $\mathbb{R}_t \times \{|x| > \varepsilon_n\}$ and $\mathbb{R}_t \times \{|x| > \varepsilon_m\}$, respectively, we obtain

$$\begin{aligned} \dot{\mathcal{E}}_{m,n}(t) &= \frac{d}{dt} \left\{ \dots \right\} - \partial_{tt} u_m(t, \varepsilon_m) \cdot \int_{\varepsilon_n < |x| < \varepsilon_m} \partial_t U_n(t, x) dx - c^2 \partial_t u_m(t, \varepsilon_m) \cdot \int_{\varepsilon_n < |x| < \varepsilon_m} \Delta U_n(t, x) dx \\ &\quad + c^2 \int_{\varepsilon_n < |x| < \varepsilon_m} (\partial_t U_n) \Delta U_n + \nabla U_n \cdot \nabla (\partial_t U_n) dx \\ &\quad + c^2 \int_{|x| > \varepsilon_m} (\partial_t [U_n - U_m]) \Delta [U_n - U_m] + \nabla [U_n - U_m] \cdot \nabla [\partial_t U_n - \partial_t U_m] dx, \end{aligned}$$

where $\{\dots\}$ denotes the curly-bracketed term in the previous expression. Integrating by parts in the last three integrals and applying the Neumann boundary condition then yield

$$\begin{aligned}
\dot{\mathcal{E}}_{m,n}(t) &= \frac{d}{dt} \left\{ \dots \right\} - \partial_{tt} u_m(t, \varepsilon_m) \cdot \int_{\varepsilon_n < |x| < \varepsilon_m} \partial_t U_n(t, x) dx \\
&\quad - c^2 \partial_t u_m(t, \varepsilon_m) \left[- \bar{c} \varepsilon_n^2 \overrightarrow{\partial_r u_n(t, \varepsilon_n)} + \bar{c} \varepsilon_m^2 \partial_r u_n(t, \varepsilon_m) \right] \\
&\quad - c^2 \bar{c} \varepsilon_n^2 \partial_t u_n(t, \varepsilon_n) \overrightarrow{\partial_r u_n(t, \varepsilon_n)} + c^2 \bar{c} \varepsilon_m^2 \partial_t u_n(t, \varepsilon_m) \partial_r u_n(t, \varepsilon_m) \\
&\quad - c^2 \bar{c} \varepsilon_m^2 \left[\partial_t u_n(t, \varepsilon_m) - \partial_t u_m(t, \varepsilon_m) \right] \cdot \left[\partial_r u_n(t, \varepsilon_m) - \overrightarrow{\partial_r u_m(t, \varepsilon_m)} \right] \\
&= \frac{d}{dt} \left\{ \dots \right\} - \partial_{tt} u_m(t, \varepsilon_m) \cdot \int_{\varepsilon_n < |x| < \varepsilon_m} \partial_t U_n(t, x) dx.
\end{aligned}$$

We then integrate back up in time, and apply integration by parts to the last integral (to avoid estimating the trace of a second derivative), to obtain

$$\begin{aligned}
\mathcal{E}_{m,n}(t) &= \mathcal{E}_{m,n}(0) + \left[\left\{ \dots \right\} \right]_0^t - \int_0^t \partial_{tt} u_m(s, \varepsilon_m) \cdot \left(\int_{\varepsilon_n < |x| < \varepsilon_m} \partial_t U_n(s, x) dx \right) ds \\
&= \mathcal{E}_{m,n}(0) + \left[\left\{ \dots \right\} \right]_0^t - \left[\partial_t u_m(s, \varepsilon_m) \cdot \left(\int_{\varepsilon_n < |x| < \varepsilon_m} \partial_t U_n(s, x) dx \right) \right]_0^t \\
&\quad + \int_0^t \partial_t u_m(s, \varepsilon_m) \cdot \left(\int_{\varepsilon_n < |x| < \varepsilon_m} \partial_{tt} U_n(s, x) dx \right) ds. \tag{3.20}
\end{aligned}$$

We next estimate the various terms in (3.20).

Step 1. First consider the trace term $\partial_t u_m(t, \varepsilon_m) = \partial_t U_m(t, \varepsilon_m \vec{e}_1)$. Set

$$W(t, x) = \partial_t U_m(t, x)$$

and observe that $W(t, x)$ solves the following initial-boundary value problem:

$$\begin{cases} \square_{1+3} W = 0 & \text{on } |x| > \varepsilon_m \\ W(0, x) = \Psi_m(x) & \text{on } |x| > \varepsilon_m \\ \partial_t W(0, x) = c^2 \Delta \Phi_m(x) & \text{on } |x| > \varepsilon_m \\ \frac{\partial W}{\partial \nu}(t, x) \equiv 0 & \text{on } |x| = \varepsilon_m. \end{cases}$$

Thus the energy $\mathcal{E}_W(t)$ is conserved and we have

$$\mathcal{E}_W(t) = \mathcal{E}_{\partial_t U_m}(t) \equiv \frac{1}{2} \int_{|x| > \varepsilon_m} |\partial_t W(0, x)|^2 + |\nabla W(0, x)|^2 dx \leq A_m^2,$$

where

$$A_m := \|\Psi_m\|_{H^1(\mathbb{R}^3)} + \|\Phi_m\|_{H^2(\mathbb{R}^3)},$$

This gives

$$\begin{aligned}
|W(t, \varepsilon_m \vec{e}_1)| &= \left| \int_{\varepsilon_m}^{\infty} \partial_r [W(t, r \vec{e}_1)] dr \right| \leq \int_{\varepsilon_m}^{\infty} |\nabla W(t, r \vec{e}_1)| dr \\
&\leq \left[\int_{\varepsilon_m}^{\infty} \frac{1}{r^2} dr \right]^{\frac{1}{2}} \left[\int_{\varepsilon_m}^{\infty} |\nabla W(t, r \vec{e}_1)|^2 r^2 dr \right]^{\frac{1}{2}} \\
&\lesssim \frac{1}{\varepsilon_m^{\frac{1}{2}}} \left[\int_{|x| > \varepsilon_m} |\nabla W(t, x)|^2 dx \right]^{\frac{1}{2}} \\
&\lesssim \frac{1}{\varepsilon_m^{\frac{1}{2}}} (\|\Psi_m\|_{H^1(\mathbb{R}^3)} + \|\Phi_m\|_{H^2(\mathbb{R}^3)}),
\end{aligned}$$

such that

$$|\partial_t u_m(t, \varepsilon_m)| \lesssim \frac{A_m}{\varepsilon_m^{\frac{1}{2}}}. \quad (3.21)$$

Below we use this estimate at any time $t \in [0, T)$ and for any index m .

Remark 3.3. *To obtain the key estimate (3.21) we need to assume that the initial data (Φ, Ψ) for (CP) belong to $H^2 \times H^1(\mathbb{R}^3)$. With an energy-based approach this is required for estimating H^1 -differences of exterior solutions at later times. This is the technical reason why we do not recover optimal regularity information about the limiting solution of (CP).*

Step 2. Now return to (3.20); repeated application of (3.21) and the Cauchy-Schwarz inequality yield

$$\begin{aligned}
\mathcal{E}_{m,n}(t) &\lesssim \mathcal{E}_{m,n}(0) + \left\{ \varepsilon_n^3 \left(\frac{A_n^2}{\varepsilon_n} + \frac{A_m^2}{\varepsilon_m} \right) + (\varepsilon_m^3 - \varepsilon_n^3) \frac{A_m^2}{\varepsilon_m} \right\} \\
&\quad + \frac{A_m}{\varepsilon_m^{\frac{1}{2}}} (\varepsilon_m^3 - \varepsilon_n^3)^{\frac{1}{2}} \left\{ \left[\int_{\varepsilon_n < |x| < \varepsilon_m} |\partial_t U_n(t, x)|^2 dx \right]^{\frac{1}{2}} + \left[\int_{\varepsilon_n < |x| < \varepsilon_m} |\partial_t U_n(0, x)|^2 dx \right]^{\frac{1}{2}} \right\} \\
&\quad + \frac{A_m}{\varepsilon_m^{\frac{1}{2}}} (\varepsilon_m^3 - \varepsilon_n^3)^{\frac{1}{2}} \int_0^t \left[\int_{\varepsilon_n < |x| < \varepsilon_m} |\partial_{tt} U_n(s, x)|^2 dx \right]^{\frac{1}{2}} ds \\
&\lesssim \mathcal{E}_{m,n}(0) + \varepsilon_n^2 A_n^2 + \varepsilon_m^2 A_m^2 + \varepsilon_m A_m A_n, \quad (3.22)
\end{aligned}$$

where we have used that A_n also bounds $\mathcal{E}_{U_n}(t)$.

Step 3. It remains to estimate the term $\mathcal{E}_{m,n}(0)$. For this we return to (3.19) and argue as above to get:

$$\begin{aligned}
\mathcal{E}_{m,n}(0) &\lesssim \varepsilon_n A_n^2 + \varepsilon_m A_m^2 + \varepsilon_m A_m A_n + \|\Psi_n\|_{L^2(\varepsilon_n < |x| < \varepsilon_m)}^2 + \|\Phi_n\|_{H^1(\varepsilon_n < |x| < \varepsilon_m)}^2 \\
&\quad + \|\Psi_n - \Psi_m\|_{L^2(|x| > \varepsilon_m)}^2 + \|\Phi_n - \Phi_m\|_{H^1(|x| > \varepsilon_m)}^2. \quad (3.23)
\end{aligned}$$

According to (3.15) and (3.16), the A_n remain bounded while (Φ_n) and (Ψ_n) are Cauchy sequences in $H^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, respectively. Using this in (3.23) and (3.22) shows that $\mathcal{E}_{m,n}(t)$ tends to zero, uniformly with respect to $t \in [0, T]$, as $m, n \rightarrow \infty$.

Step 4. Finally we consider the L^2 -distance between $\tilde{U}_n(t)$ and $\tilde{U}_m(t)$. For this set

$$\mathcal{D}_{m,n}(t) := \frac{1}{2} \int_{\mathbb{R}^3} |Z_{m,n}(t, x)|^2 dx \quad (Z_{m,n} := \tilde{U}_n - \tilde{U}_m)$$

and observe that the Cauchy-Schwarz inequality yields

$$\dot{\mathcal{D}}_{m,n}(t) \leq 2\mathcal{D}_{m,n}(t)^{\frac{1}{2}} \mathcal{E}_{m,n}(t)^{\frac{1}{2}}.$$

Thus,

$$\mathcal{D}_{m,n}(t) \lesssim \mathcal{D}_{m,n}(0) + \sup_{0 \leq s \leq T} \mathcal{E}_{m,n}(s).$$

Since

$$\mathcal{D}_{m,n}(0) \lesssim \|\Phi_n - \Phi_m\|_{L^2(\mathbb{R}^3)} \rightarrow 0 \quad \text{as } m, n \rightarrow \infty,$$

we conclude that $\mathcal{D}_{m,n}(t)$ tends to zero, uniformly with respect to $t \in [0, T]$, as $m, n \rightarrow \infty$.

It follows from the estimates above that $\|\tilde{U}_n(t) - \tilde{U}_m(t)\|_{H^1(\mathbb{R}^3)}$ tends to zero as $m, n \rightarrow \infty$, uniformly with respect to $t \in [0, T]$.

3.3.3. Continuity in time of extended exterior solution. We next verify that the extended exterior Neumann solutions \tilde{U}_n belong to $C([0, T]; H^1(\mathbb{R}^3))$. Indeed, for $0 \leq s \leq t < T$ we have by Cauchy-Schwarz that

$$\begin{aligned} \|\tilde{U}_n(t) - \tilde{U}_n(s)\|_{H^1(\mathbb{R}^3)}^2 &= \int_{\mathbb{R}^3} |\tilde{U}_n(t, x) - \tilde{U}_n(s, x)|^2 + |\nabla \tilde{U}_n(t, x) - \nabla \tilde{U}_n(s, x)|^2 dx \\ &\leq (t-s) \int_{\mathbb{R}^3} \int_s^t |\partial_t \tilde{U}_n(\tau, x)|^2 + |\partial_t \nabla \tilde{U}_n(\tau, x)|^2 d\tau dx \\ &\lesssim (t-s)^2 \cdot \sup_{0 \leq \tau \leq T} \mathbf{E}_n(\tau), \end{aligned} \tag{3.24}$$

where

$$\mathbf{E}_n(\tau) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t \tilde{U}_n(\tau, x)|^2 + |\partial_t \nabla \tilde{U}_n(\tau, x)|^2 dx.$$

We now have

$$\mathbf{E}_n(\tau) \lesssim \mathcal{E}_n(\tau) + U_n(\tau),$$

where

$$\mathcal{E}_n(\tau) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t \tilde{U}_n(\tau, x)|^2 + c^2 |\nabla \tilde{U}_n(\tau, x)|^2 dx,$$

and $U_n(\tau)$ was defined in Section 3.3.1. As observed there, $U_n(\tau)$ is constant in time and it follows that

$$\sup_{0 \leq \tau \leq T} U_n(\tau) \lesssim \|\Phi_n\|_{H^2(\mathbb{R}^3)} + \|\Psi_n\|_{H^1(\mathbb{R}^3)},$$

which is uniformly bounded according to (3.15)-(3.16). Finally, to bound $\mathcal{E}_n(\tau)$ we observe that it coincides with $\mathcal{E}_{m,n}(\tau)$ (see (3.17)) in the special case that Φ_m and Ψ_m are zero (such that \tilde{U}_m vanishes identically). Therefore, the estimates in Section 3.3.2 show that $\mathcal{E}_n(\tau)$ is uniformly bounded for $\tau \in [0, T]$. We conclude that each \tilde{U}_n maps $[0, T]$ Lipschitz continuously into $H^1(\mathbb{R}^3)$, and that their Lipschitz constants is uniformly bounded with respect to n .

3.3.4. *Weak H^1 -solution as the limit of exterior Neumann solutions.* The argument in Section 3.3.2 showed that

$$\sup_{0 \leq t < T} \|\tilde{U}_n(t) - \tilde{U}_m(t)\|_{H^1(\mathbb{R}^3)} \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

By the argument in Section 3.3.3, (\tilde{U}_n) is a Cauchy sequence in $C([0, T]; H^1(\mathbb{R}^3))$. We let U denote its limit, and we claim that it is a weak H^1 -solution to the original Cauchy problem (CP) according to Definition 2.1. For this, fix $V \in C_c^\infty((-\infty, T) \times \mathbb{R}^3)$; as each \tilde{U}_n solves the 3-d wave equation with Neumann conditions along $\{|x| = \varepsilon_n\}$, we have that

$$\int_0^T \int_{|x| \geq \varepsilon_n} U_n V_{tt} + c^2 \nabla U_n \cdot \nabla V \, dx dt + \int_{|x| \geq \varepsilon_n} \Phi_n(x) V_t(0, x) - \Psi_n(x) V(0, x) \, dx = 0,$$

for each n . It is now routine to apply the estimates in Section 3.3.2 to show that each term on the left hand side tends to the corresponding term in (2.1) as $n \rightarrow \infty$. E.g., for the first term we have

$$\begin{aligned} & \left| \int_0^T \int_{|x| \geq \varepsilon_n} U_n V_{tt} \, dx dt - \int_0^T \int_{\mathbb{R}^3} U V_{tt} \, dx dt \right| \\ & \leq \int_0^T \int_{|x| \geq \varepsilon_n} |U_n - U| |V_{tt}| \, dx dt + \int_0^T \int_{|x| \leq \varepsilon_n} |U| |V_{tt}| \, dx dt \\ & \lesssim \sup_{0 \leq t < T} \|\tilde{U}_n(t) - U(t)\|_{L^2(\mathbb{R}^3)} + \varepsilon_n^{\frac{3}{2}} \cdot \sup_{0 \leq t < T} \|U(t)\|_{L^2(\mathbb{R}^3)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The other terms are treated similarly. This concludes the proof of part (i) of Theorem 2.3.

Remark 3.4. *It is natural to ask if the same approach can be applied when the initial data for (CP) belong to $H^1 \times L^2(\mathbb{R}^3)$. However, we have not been able to establish convergence of exterior Neumann solutions in this case. As noted in Remark 3.3, we make essential use of $H^2 \times H^1(\mathbb{R}^3)$ -regularity of the Cauchy data to estimate the energy of the difference between two exterior solutions.*

4. CAUCHY SOLUTION AS LIMIT OF EXTERIOR DIRICHLET SOLUTIONS: WEAK L^2 -SOLUTION FOR $H^2 \times H^1$ -DATA VIA COMPACTNESS

In this section we construct exterior Dirichlet solutions and study their convergence toward a weak solution of (CP). In contrast to the case of exterior Neumann solutions, we now apply a compactness argument.

4.1. **Exterior Dirichlet data and solutions.** We employ the following scheme for exterior Dirichlet solutions approximating the solution of the Cauchy problem (CP) with initial data $(\Phi, \Psi) \in H_{rad}^2 \times H_{rad}^1(\mathbb{R}^3)$.

- (1) As in Section 3.1 we first fix sequences $\bar{\Phi}_n, \bar{\Psi}_n$ in $C_{c,rad}^\infty(\mathbb{R}^3)$ such that

$$\bar{\Phi}_n \rightarrow \Phi \quad \text{in } H^2(\mathbb{R}^3) \quad \text{and} \quad \bar{\Psi}_n \rightarrow \Psi \quad \text{in } H^1(\mathbb{R}^3) \quad \text{as } n \rightarrow \infty. \quad (4.1)$$

- (2) We consider any sequence (ε_n) of radii with $\varepsilon_n \downarrow 0$. To smoothly approximate the original data (Φ, Ψ) with exterior Dirichlet data we fix a smooth, nondecreasing function $\chi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ with

$$\chi \equiv 0 \quad \text{on } [0, 1], \quad \chi \equiv 1 \quad \text{on } [2, \infty). \quad (4.2)$$

Then, with

$$\bar{\varphi}_n(|x|) := \bar{\Phi}_n(x) \quad \text{and} \quad \bar{\psi}_n(|x|) := \bar{\Psi}_n(x),$$

we define

$$\Phi_n(x) := \bar{\varphi}_n(|x|)\chi\left(\frac{|x|}{\varepsilon_n}\right) \quad \text{and} \quad \Psi_n(x) := \bar{\psi}_n(|x|)\chi\left(\frac{|x|}{\varepsilon_n}\right). \quad (4.3)$$

For later reference we set

$$\varphi_n(r) := \Phi_n(r\vec{e}_1) \quad \text{and} \quad \psi_n(r) := \Psi_n(r\vec{e}_1). \quad (4.4)$$

Note that the restrictions of both Φ_n and Ψ_n to the exterior domain

$$\Omega_n := \{|x| \geq \varepsilon_n\}.$$

satisfy homogeneous Dirichlet conditions at $|x| = \varepsilon_n$.

We refer to (Φ_n, Ψ_n) as the *Dirichlet data* corresponding to the original Cauchy data (Φ, Ψ) for (CP). Note that the Dirichlet data are defined on all of \mathbb{R}^3 . We analyze their convergence to (Φ, Ψ) in Section 4.2 below.

- (3) It is standard that the exterior Dirichlet problem on Ω_n with data $(\Phi_n|_{\Omega_n}, \Psi_n|_{\Omega_n})$ has a unique, smooth, and global-in-time solution which we denote by $U_n(t, x)$. This may be established via solution formulae based on d'Alembert's formula for the 1-d linear wave equation, [1, 14]. To compare these we extend each $U_n(t, x)$ continuously as zero at each time:

$$\tilde{U}_n(t, x) := \begin{cases} 0 & \text{for } |x| \leq \varepsilon_n \\ U_n(t, x) & \text{for } |x| \geq \varepsilon_n, \end{cases} \quad (4.5)$$

Note that, differently from in the Neumann case, the extended solution \tilde{U}_n defined in (4.5) will not be C^1 -smooth across $|x| = \varepsilon_n$.

4.2. Dirichlet data vs. Cauchy data. We next consider how the Dirichlet data (Φ_n, Ψ_n) , which are defined on all of \mathbb{R}^3 , approximate the original Cauchy data $(\Phi, \Psi) \in H^2 \times H^1(\mathbb{R}^3)$. This information is used later to establish compactness of the exterior Dirichlet solutions. Specifically, we want to estimate

$$\|\Phi_n - \Phi\|_{H^2(\mathbb{R}^3)} \quad \text{and} \quad \|\Psi_n - \Psi\|_{H^1(\mathbb{R}^3)}.$$

Thanks to (4.1) it suffices to estimate

$$\|\Phi_n - \bar{\Phi}_n\|_{H^2(\mathbb{R}^3)} \quad \text{and} \quad \|\Psi_n - \bar{\Psi}_n\|_{H^1(\mathbb{R}^3)}.$$

To lighten the notation we consider, for now, a fixed function $\bar{F} \in C_{c,rad}^\infty(\mathbb{R}^3)$, say

$$\bar{F}(x) = \bar{f}(|x|),$$

and set

$$F_\varepsilon(x) = f_\varepsilon(|x|) := \bar{f}(|x|)\chi\left(\frac{|x|}{\varepsilon}\right), \quad \varepsilon > 0. \quad (4.6)$$

Thus, \bar{F} corresponds to $\bar{\Phi}_n$ or $\bar{\Psi}_n$, F_ε corresponds to Φ_n or Ψ_n , respectively, and ε corresponds to ε_n .

For a fixed value of $\varepsilon > 0$ we then proceed to estimate $\|F_\varepsilon - \bar{F}\|_{L^2(\mathbb{R}^3)}$, $\|\nabla F_\varepsilon - \nabla \bar{F}\|_{L^2(\mathbb{R}^3)}$, and $\|D^2 F_\varepsilon - D^2 \bar{F}\|_{L^2(\mathbb{R}^3)}$ ($1 \leq i, j \leq 3$). The following estimates illustrate the fact that approximating Cauchy data with exterior Dirichlet data introduces larger errors than with Neumann data. Recall that the difference between Cauchy and Neumann data converges to

zero in $H^2(\mathbb{R}^3)$, cf. (3.15). In contrast, we show that, while the difference between Cauchy and Dirichlet data tends to zero in $H^1(\mathbb{R}^3)$, it may be unbounded in $H^2(\mathbb{R}^3)$.

4.2.1. $\|F_\varepsilon - \bar{F}\|_{L^2(\mathbb{R}^3)}$. From (4.6) we obtain

$$\|F_\varepsilon - \bar{F}\|_{L^2(\mathbb{R}^3)}^2 = \int_{|x| < 2\varepsilon} |\bar{F}(x)(\chi(\frac{|x|}{\varepsilon}) - 1)|^2 dx \leq \|\bar{F}\|_{L^2(B_{2\varepsilon})}^2. \quad (4.7)$$

4.2.2. $\|\nabla F_\varepsilon - \nabla \bar{F}\|_{L^2(\mathbb{R}^3)}$. In a similar way we obtain from (3.7) that

$$\begin{aligned} \sum_i \|\partial_i F_\varepsilon - \partial_i \bar{F}\|_{L^2(\mathbb{R}^3)}^2 &= \int_{|x| < 2\varepsilon} \left| \bar{F}'(|x|)(\chi(\frac{|x|}{\varepsilon}) - 1) + \frac{1}{\varepsilon} \bar{F}(|x|)\chi'(\frac{|x|}{\varepsilon}) \right|^2 dx \\ &\lesssim \int_{|x| < 2\varepsilon} |\bar{F}'(|x|)|^2 dx + \frac{1}{\varepsilon^2} \int_{\varepsilon < |x| < 2\varepsilon} |\bar{F}(|x|)|^2 dx. \end{aligned} \quad (4.8)$$

We estimate the last term by reducing to 1-d, applying the Cauchy-Schwarz inequality, and finally applying Hardy's inequality (2.3):

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{\varepsilon < |x| < 2\varepsilon} |\bar{F}(|x|)|^2 dx &\lesssim \frac{1}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} |\bar{F}(r)|^2 r^2 dr = \frac{1}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} \left| -2 \int_r^\infty \bar{F}(s) \bar{F}'(s) ds \right|^2 r^2 dr \\ &\lesssim \frac{1}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} \left[\int_r^\infty |\bar{F}(s)|^2 ds \right]^{\frac{1}{2}} \left[\int_r^\infty |\bar{F}'(s)|^2 ds \right]^{\frac{1}{2}} r^2 dr \\ &= \frac{1}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} \left[\int_r^\infty \frac{|\bar{F}(s)|^2}{s^2} s^2 ds \right]^{\frac{1}{2}} \left[\int_r^\infty \frac{|\bar{F}'(s)|^2}{s^2} s^2 ds \right]^{\frac{1}{2}} r^2 dr \\ &\lesssim \frac{1}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} \left[\int_{|x| > r} \frac{|\bar{F}(x)|^2}{|x|^2} dx \right]^{\frac{1}{2}} \left[\int_{|x| > r} \frac{|\nabla \bar{F}(x)|^2}{|x|^2} dx \right]^{\frac{1}{2}} r^2 dr \\ &\lesssim \varepsilon \left\| \frac{\bar{F}}{|x|} \right\|_{L^2(\mathbb{R}^3)} \left\| \frac{\nabla \bar{F}}{|x|} \right\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \varepsilon \|\bar{F}\|_{H^1(\mathbb{R}^3)} \|\bar{F}\|_{H^2(\mathbb{R}^3)} \lesssim \varepsilon \|\bar{F}\|_{H^2(\mathbb{R}^3)}^2. \end{aligned} \quad (4.9)$$

Combining this with (4.8) we get that

$$\sum_i \|\partial_i F_\varepsilon - \partial_i \bar{F}\|_{L^2(\mathbb{R}^3)}^2 \lesssim \|\bar{F}\|_{H^1(B_{2\varepsilon})}^2 + \varepsilon \|\bar{F}\|_{H^2(\mathbb{R}^3)}^2. \quad (4.10)$$

Remark 4.1. The fact that $F_\varepsilon \rightarrow \bar{F}$ in $H^1(\mathbb{R}^3)$ as $\varepsilon \downarrow 0$ is a special case of the more general result that the set of $C_c^\infty(\mathbb{R}^n)$ -functions vanishing on balls about a point is dense in $H^1(\mathbb{R}^n)$ for $n \geq 2$; see Lemma 17.2 and Lemma 17.3 in [18].

4.2.3. $\|D^2 F_\varepsilon - D^2 \bar{F}\|_{L^2(\mathbb{R}^3)}$. We proceed to analyze the L^2 -distance between second derivatives of the given Cauchy data and their corresponding Dirichlet data. While this may blow up as $\varepsilon \downarrow 0$, we still obtain a useful estimate. First, applying (3.8) to $F_\varepsilon - \bar{F}$ yields

$$\begin{aligned} \sum_{i,j} |\partial_{ij} F_\varepsilon(x) - \partial_{ij} \bar{F}(x)|^2 &= \left| \bar{F}''(|x|)(\chi(\frac{|x|}{\varepsilon}) - 1) + \frac{2}{\varepsilon} \bar{F}'(|x|)\chi'(\frac{|x|}{\varepsilon}) + \frac{1}{\varepsilon^2} \bar{F}(|x|)\chi''(\frac{|x|}{\varepsilon}) \right|^2 \\ &\quad + \frac{2}{|x|^2} \left| \bar{F}'(|x|)(\chi(\frac{|x|}{\varepsilon}) - 1) + \frac{1}{\varepsilon} \bar{F}(|x|)\chi'(\frac{|x|}{\varepsilon}) \right|^2. \end{aligned} \quad (4.11)$$

It follows that

$$\begin{aligned} \sum_{i,j} \|\partial_{ij} F_\varepsilon - \partial_{ij} \bar{F}\|_{L^2(\mathbb{R}^3)}^2 &\lesssim \int_{|x| < 2\varepsilon} |\bar{f}''(|x|)|^2 + \frac{1}{|x|^2} |\bar{f}'(|x|)|^2 dx \\ &\quad + \int_{\varepsilon < |x| < 2\varepsilon} \frac{1}{\varepsilon^2} |\bar{f}'(|x|)|^2 + \frac{1}{\varepsilon^4} |\bar{f}(|x|)|^2 dx \\ &\lesssim \int_{|x| < 2\varepsilon} |\bar{f}''(|x|)|^2 + \frac{1}{|x|^2} |\bar{f}'(|x|)|^2 dx + \frac{1}{\varepsilon^4} \int_{\varepsilon < |x| < 2\varepsilon} |\bar{f}(|x|)|^2 dx. \end{aligned}$$

We use (3.8) in the first integral and (4.9) in the second integral to deduce that

$$\sum_{i,j} \|\partial_{ij} F_\varepsilon - \partial_{ij} \bar{F}\|_{L^2(\mathbb{R}^3)}^2 \lesssim \|\bar{F}\|_{H^2(B_{2\varepsilon})}^2 + \frac{1}{\varepsilon} \|\bar{F}\|_{H^2(\mathbb{R}^3)}^2. \quad (4.12)$$

We finally apply the bounds (4.7), (4.10), and (4.12) with $(\bar{F}, F_\varepsilon) = (\bar{\Phi}_n, \Phi_n)$ and $(\bar{F}, F_\varepsilon) = (\bar{\Psi}_n, \Psi_n)$. Recalling (4.1) we conclude that the Dirichlet data (Φ_n, Ψ_n) corresponding to the original Cauchy data $(\Phi, \Psi) \in H^2 \times H^1(\mathbb{R}^3)$ for (CP), satisfy

$$\|\Phi_n - \Phi\|_{H^1(\mathbb{R}^3)} \rightarrow 0, \quad (4.13)$$

$$\|\Phi_n - \Phi\|_{H^2(\mathbb{R}^3)} \lesssim \frac{1}{\varepsilon_n^{\frac{1}{2}}}, \quad (4.14)$$

$$\|\Psi_n - \Psi\|_{H^1(\mathbb{R}^3)} \rightarrow 0. \quad (4.15)$$

Remark 4.2. We remark that (4.14) reflects the general fact that the set

$$\{F \in C_c^\infty(\mathbb{R}^3) \mid F \equiv 0 \text{ on } B_r \text{ for some } r > 0\}$$

is not dense in $H^2(\mathbb{R}^3)$. As we are not aware of a reference for this, we include a sketch of a proof. First, the issue concerns the local behavior near the origin, and we can restrict attention to the unit ball $B_1 \subset \mathbb{R}^3$. Next, since replacing a function by the corresponding spherically averaged function does not increase H^k -norms, it suffices to argue that the set

$$\mathcal{X} := \{F \in C_{rad}^\infty(B_1) \mid F \equiv 0 \text{ on } B_r \text{ for some } r > 0\}$$

is not dense in $H^2(B_1)$. Assume for contradiction that there is a sequence $(F_n) \subset \mathcal{X}$, with F_n vanishing on B_{r_n} , and such that $F_n \rightarrow F \equiv 1$ in $H_{rad}^2(B_1)$. It follows that there is a subsequence, still denoted (F_n) , such that $F_n(x) \rightarrow 1$ for almost all $x \in B_1$; let \bar{x} be such a point and set $\bar{r} = |\bar{x}|$. With $F_n(x) := f_n(|x|)$ Cauchy-Schwarz gives

$$|f_n(\bar{r})|^2 = |f_n(\bar{r}) - f_n(r_n)|^2 \leq \int_0^1 |f_n'(r)|^2 dr. \quad (4.16)$$

According to Hardy's inequality (2.4), applied to ∇F_n , we have

$$\int_0^1 |f_n'(r)|^2 dr \lesssim \int_{B_1} \frac{|\nabla F_n(x)|^2}{|x|^2} dx \lesssim \int_{B_1} |\nabla F_n(x)|^2 + |D^2 F_n(x)|^2 dx,$$

which tends to zero as $n \rightarrow \infty$, since $F_n \rightarrow 1$ in $H_{rad}^2(B_1)$. But then (4.16) gives $f_n(\bar{r}) \rightarrow 0$, contradicting the choice of \bar{x} .

4.3. Convergence of exterior Dirichlet solutions via compactness. We next want to establish that the extended Dirichlet solutions \tilde{U}_n converge to a weak L^2 -solution of the original Cauchy problem (CP) according to Definition 2.2.

As detailed below, we prove convergence of a subsequence by establishing relative compactness of (\tilde{U}_n) in $C([0, T]; L^2(\mathbb{R}^3))$. We note that this compactness argument applies whenever the data for (CP) belong to $H^1 \times L^2(\mathbb{R}^3)$. However, as a manifestation of the lower regularity of exterior Dirichlet solutions as compared to exterior Neumann solutions, we are only able to show that the corresponding limit is a weak L^2 -solution when the data for the original Cauchy problem (CP) belong to $H^2 \times H^1(\mathbb{R}^3)$; see Remark 4.3. (And even for such data we need to exploit a special property of the linear 3-d wave equation, see subsection 4.3.2.)

4.3.1. Compactness in $C([0, T]; L^2(\mathbb{R}^3))$. To show that (\tilde{U}_n) is relatively compact in the space $C([0, T]; L^2(\mathbb{R}^3))$ it suffices, according to Lemma 1 in [17], to establish relative compactness of $(\tilde{U}_n(t))$ in $L^2(\mathbb{R}^3)$ at each time $t \in [0, T)$, together with uniform equicontinuity of the maps $t \mapsto \tilde{U}_n(t)$. First, according to the Kolmogorov-Riesz-Fréchet theorem [2] the first issue amounts to showing that for each fixed $t \in [0, T)$:

- (a) $(\tilde{U}_n(t))$ is bounded in $L^2(\mathbb{R}^3)$;
- (b) for each $\varepsilon > 0$ there is a $\rho = \rho(\varepsilon) > 0$ such that, independently of n ,

$$\int_{\mathbb{R}^3} |\tilde{U}_n(t, x+h) - \tilde{U}_n(t, x)|^2 dx < \varepsilon^2 \quad \text{whenever } |h| < \rho;$$

- (c) for each $\varepsilon > 0$ there is an $R = R(\varepsilon) > 0$ such that, independently of n ,

$$\int_{\{|x|>R\}} |\tilde{U}_n(t, x)|^2 dx < \varepsilon^2.$$

Secondly, for equicontinuity we shall establish uniform Lipschitz continuity of $t \mapsto \tilde{U}_n(t)$. We first consider (a)-(c) and introduce the energy

$$\mathcal{E}_n(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t \tilde{U}_n(t, x)|^2 + c^2 |\nabla \tilde{U}_n(t, x)|^2 dx. \quad (4.17)$$

For the Dirichlet case the extensions \tilde{U}_n vanish identically on B_{ε_n} , and we have

$$\mathcal{E}_n(t) = \frac{1}{2} \int_{|x|>\varepsilon_n} |\partial_t U_n(t, x)|^2 + c^2 |\nabla U_n(t, x)|^2 dx, \quad (4.18)$$

which is conserved in time. We also set

$$\mathcal{D}_n(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\tilde{U}_n(t, x)|^2 dx = \frac{1}{2} \int_{|x|>\varepsilon_n} |U_n(t, x)|^2 dx. \quad (4.19)$$

To verify (a) we consider $\dot{\mathcal{D}}_n(t)$ and apply Cauchy-Schwarz to obtain (cf. the estimation of $\dot{\mathcal{D}}_{m,n}(t)$ in Section 3.3.2)

$$\frac{d}{dt} [\mathcal{D}_n(t)]^{\frac{1}{2}} \lesssim [\mathcal{E}_n(t)]^{\frac{1}{2}} = [\mathcal{E}_n(0)]^{\frac{1}{2}} \lesssim \|\Phi_n\|_{H^1(\mathbb{R}^3)} + \|\Psi_n\|_{L^2(\mathbb{R}^3)},$$

which is uniformly bounded with respect to n according to (4.13) and (4.15). Also, $\mathcal{D}_n(0)$ is uniformly bounded according to (4.13). It follows that $\mathcal{D}_n(t)$ is uniformly bounded with respect to both n and $t \in [0, T)$.

To verify (b) we apply Cauchy-Schwarz in a standard manner to get that for any $h \in \mathbb{R}^3$,

$$\begin{aligned} \int_{\mathbb{R}^3} |\tilde{U}_n(t, x+h) - \tilde{U}_n(t, x)|^2 dx &\leq |h|^2 \int_{\mathbb{R}^3} \int_0^1 \left| \nabla \tilde{U}_n(t, x + \theta h) \right|^2 d\theta dx \\ &\lesssim |h|^2 \mathcal{E}_n(t) = |h|^2 \mathcal{E}_n(0). \end{aligned}$$

It follows from (4.13) and (4.15) that (b) holds.

Finally, to verify (c) we consider R_0 and n such that $R_0 > 1 > 2\varepsilon_n$ and define

$$\hat{\mathcal{E}}_n(t) := \frac{1}{2} \int_{|x| > R_0 + ct} |\partial_t U_n(t, x)|^2 + c^2 |\nabla U_n(t, x)|^2 dx,$$

and

$$\hat{\mathcal{D}}_n(t) := \frac{1}{2} \int_{|x| > R_0 + ct} |U_n(t, x)|^2 dx.$$

Standard calculations show that (cf. Section 2.4.3 in [4])

$$\frac{d}{dt} \hat{\mathcal{E}}_n(t) \leq 0, \quad \frac{d}{dt} [\hat{\mathcal{D}}_n(t)]^{\frac{1}{2}} \lesssim [\hat{\mathcal{E}}_n(t)]^{\frac{1}{2}} \leq [\hat{\mathcal{E}}_n(0)]^{\frac{1}{2}}.$$

Combining these bounds with (3.2) shows that

$$\int_{|x| > R_0 + ct} |U_n(t, x)|^2 dx \lesssim \|\Psi\|_{L^2(|x| > R_0 - 1)}^2 + \|\Phi\|_{H^1(|x| > R_0 - 1)}^2$$

for all n sufficiently large. Since the right hand side decreases to zero as $R_0 \rightarrow \infty$, we have verified (c).

We conclude from the Kolmogorov-Riesz-Fréchet theorem for $L^2(\mathbb{R}^3)$ that $(\tilde{U}_n(t))$ is relatively compact in $L^2(\mathbb{R}^3)$ at each time $t \in [0, T]$. Next, a calculation similar to that in (3.24) shows that the maps $t \mapsto \tilde{U}_n(t) \in L^2(\mathbb{R}^3)$ are Lipschitz continuous, with a Lipschitz constant that is bounded uniformly with respect to n . It follows from Lemma 1 in [17] that there is a subsequence of (\tilde{U}_n) , still denoted (\tilde{U}_n) , and a $U \in C([0, T]; L^2(\mathbb{R}^3))$ such that

$$\tilde{U}_n \rightarrow U \quad \text{in } C([0, T]; L^2(\mathbb{R}^3)). \quad (4.20)$$

Remark 4.3. We note that the possible blowup of the H^2 -norm of the exterior Dirichlet data prevents us from repeating the same argument to the first order derivatives of \tilde{U}_n . Thus, in the case of exterior Dirichlet solutions, we do not obtain a candidate for a weak H^1 -solution by arguing via compactness.

4.3.2. Weak L^2 -solution as the limit of exterior Dirichlet solutions for $H^2 \times H^1(\mathbb{R}^3)$ -data. It remains to verify that the limit $U \in C([0, T]; L^2(\mathbb{R}^3))$ obtained above is indeed a weak L^2 -solution of the original Cauchy problem (CP) according to Definition 2.2.

As each exterior solution U_n is a classical solution of the wave equation on $\{|x| > \varepsilon_n\}$ and satisfies the Dirichlet condition $U_n(t, x) \equiv 0$ for $|x| = \varepsilon_n$, we have

$$\begin{aligned} \int_0^T \int_{|x| \geq \varepsilon_n} U_n \square_{1+3} V dx dt + \int_{|x| \geq \varepsilon_n} V_t(0, x) \Phi_n(x) - V(0, x) \Psi_n(x) dx \\ = -c^2 \int_0^T \partial_r U_n(t, \varepsilon_n \vec{e}_1) \left(\int_{|x| = \varepsilon_n} V(t, x) dS_x \right) dt, \end{aligned} \quad (4.21)$$

for each n , whenever $V \in C_c^\infty((-\infty, T) \times \mathbb{R}^3)$. (Here $\partial_r U_n(t, x) = \nabla U_n(t, x) \cdot \frac{x}{|x|}$ denotes the radial derivative of the function U_n .) It follows from (4.20), together with (4.13) and

(4.15), that each term on the left hand side of (4.21) converges to the corresponding term in (2.2).

It remains to show that the residual on the right hand side of (4.21) tends to zero as $n \rightarrow \infty$. We shall see that this is indeed the case. However, because we insist on arguments based on energy estimates, we have been able to establish this only when the initial data for the original Cauchy problem (CP) belong to $H^2 \times H^1(\mathbb{R}^3)$ (i.e., one degree more regularity than what was required for convergence in Section 4.3.1 above). As noted earlier, even for such data we find it necessary to exploit the fact that the PDE in question is the linear 3-d wave equation.

Specifically, we shall use the fact that if U is a radial solution, then $z(t, r) := rU(t, r\vec{e}_1)$ and its derivatives satisfy suitable energy estimates. This is satisfied for the 3-d wave equation since in this case $z(t, r)$ is a solution of the 1-d wave equation. Although there might not be an exact analogue of this for other (e.g. nonlinear) equations, our argument will avoid any use of explicit solution formulae.

For the exterior radial Dirichlet solution U_n we set

$$u_n(t, r) := U_n(t, r\vec{e}_1) \quad \text{and} \quad z_n(t, r) := ru_n(t, r),$$

and use the Dirichlet condition at $r = \varepsilon_n$ together with Cauchy-Schwarz, to obtain

$$\begin{aligned} |\varepsilon_n \partial_r U_n(t, \varepsilon_n \vec{e}_1)| &= |\varepsilon_n \partial_r u_n(t, \varepsilon_n)| = |\partial_r z_n(t, \varepsilon_n)| = \left[-2 \int_{\varepsilon_n}^{\infty} \partial_r z_n(t, r) \partial_{rr} z_n(t, r) dr \right]^{\frac{1}{2}} \\ &\lesssim \left[\int_{\varepsilon_n}^{\infty} |\partial_r z_n(t, r)|^2 dr \right]^{\frac{1}{4}} \left[\int_{\varepsilon_n}^{\infty} |\partial_{rr} z_n(t, r)|^2 dr \right]^{\frac{1}{4}} =: C_n^{\frac{1}{4}} D_n^{\frac{1}{4}}. \end{aligned} \quad (4.22)$$

To estimate C_n and D_n we introduce $w_n := \partial_r z_n$, and note that both z_n and w_n are solutions to the 1-d wave equation on (ε_n, ∞) . Recalling (4.4) we have that z_n has initial data $(r\varphi_n(r), r\psi_n(r))$ and satisfies the Dirichlet condition $z_n(t, \varepsilon_n) \equiv 0$, while w_n has initial data $(\partial_r(r\varphi_n(r)), \partial_r(r\psi_n(r)))$ and satisfies the Neumann condition $\partial_r w_n(t, \varepsilon_n) \equiv 0$. It follows that the energies

$$\int_{\varepsilon_n}^{\infty} |\partial_t z_n(t, r)|^2 + c^2 |\partial_r z_n(t, r)|^2 dr \quad \text{and} \quad \int_{\varepsilon_n}^{\infty} |\partial_t w_n(t, r)|^2 + c^2 |\partial_r w_n(t, r)|^2 dr$$

are both constant in time. The first of these bounds C_n such that

$$\begin{aligned} C_n &\lesssim \int_{\varepsilon_n}^{\infty} |r\psi_n(r)|^2 + |\partial_r(r\varphi_n(r))|^2 dr \\ &\lesssim \int_{|x| \geq \varepsilon_n} |\Psi_n(x)|^2 + \frac{|\Phi_n(x)|^2}{|x|^2} + |\nabla \Phi_n(x)|^2 dx. \end{aligned}$$

Recalling that Φ_n vanishes on B_{ε_n} , we obtain from Hardy's inequality (2.3) that

$$\int_{|x| \geq \varepsilon_n} \frac{|\Phi_n(x)|^2}{|x|^2} \lesssim \int_{|x| \geq \varepsilon_n} |\nabla \Phi_n(x)|^2 dx.$$

Thus,

$$C_n \lesssim \|\Phi_n\|_{H^1(\mathbb{R}^3)}^2 + \|\Psi_n\|_{L^2(\mathbb{R}^3)}^2,$$

which is uniformly bounded according to (4.13) and (4.15). For D_n we apply Hardy's inequality (2.3) to Ψ_n , and also formula (3.8) to Φ_n , to get that

$$\begin{aligned} D_n &\lesssim \int_{\varepsilon_n}^{\infty} |\partial_r(r\psi_n(r))|^2 + |\partial_{rr}(r\varphi_n(r))|^2 dr \\ &\lesssim \int_{\varepsilon_n}^{\infty} \left(|\psi'_n(r)|^2 + \frac{1}{r^2} |\psi_n(r)|^2 + |\varphi''_n(r)|^2 + \frac{2}{r^2} |\varphi'_n(r)|^2 \right) r^2 dr \\ &\lesssim \int_{|x| \geq \varepsilon_n} |\nabla \Psi_n(x)|^2 + \frac{|\Psi_n(x)|^2}{|x|^2} + \sum_{i,j} |\partial_{ij} \Phi_n(x)|^2 dx \\ &\lesssim \|\Psi_n\|_{H^1(\mathbb{R}^3)}^2 + \|\Phi_n\|_{H^2(\mathbb{R}^3)}^2. \end{aligned}$$

According to (4.14) and (4.15) we thus get that

$$D_n \lesssim \frac{1}{\varepsilon_n}.$$

Using these estimates in (4.22) we conclude that

$$\left| \int_0^T \partial_r U_n(t, \varepsilon_n \vec{e}_1) \left(\int_{|x|=\varepsilon_n} V(t, x) dS_x \right) dt \right| \lesssim \varepsilon_n^2 \int_0^T |\partial_r u_n(t, \varepsilon_n)| dt \lesssim \varepsilon_n^{\frac{3}{4}}, \quad (4.23)$$

such that the residual on the right hand side of (4.21) tends to zero as $n \rightarrow \infty$. This shows that the exterior Dirichlet solutions U_n , when extended as zero on the interior of the ball B_{ε_n} , converge to a weak L^2 -solution of the Cauchy problem (CP) according to Definition 2.2, whenever the initial data for (CP) belong to $H^2 \times H^1(\mathbb{R}^3)$. This completes the proof of part (ii) of Theorem 2.3.

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